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On the vorticity of a rotating mixture

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The centrifugal separation of an initially homogeneous mixture of particles and fluid is considered in 'long' containers for which end-wall effects are negligible. The vorticity produced in the mixture region during separation is shown to be a function of time only. As a consequence, the 'full' nonlinear theory in such containers, of arbitrary cross-section, can be reduced to the determination of an appropriate analytic function. A problem of technological interest is discussed.

1. Introduction

The theory for (centrifugal) separation of a fluid mixture is already extremely, almost prohibitively complex, although in important respects it is still incomplete and under development. Fundamental issues, both physical and mathematical in nature, remain to be resolved and among many noteworthy questions are those pertaining to the rheology of a non-dilute (rotating) mixture and the overall applicability of the continuum approximation for two-phase flows. There is a dearth of relevant experimental data and as yet no general numerical program for the motion of a viscous, incompressible mixture that is of proven accuracy and value. However, as recent work amply demonstrates (Acrivos & Herbolzheimer 1979; Greenspan & Ungarish 1985; Schneider 1982; Schaflinger, Köppl & Filipczak 1986), the present diffusion or mixture model can be used quite effectively to analyse difficult problems. The flow patterns predicted are often unexpected – involving complicated boundary layers, currents and kinematic shocks – but the results seem in good qualitative agreement with observations. This encouraging corroboration warrants the further development of the model to include mechanisms that characterize the non-dilute mixture, a distribution of particle sizes, and a description of particle collisions and cluster formation. Exact and physically relevant solutions of the equations of motion would be invaluable in this effort as a source of insight, a means to assess various processes, to craft simpler theories and ultimately to test the numerical programs that are forthcoming. Solutions for gravitational and centrifugal settling (Kynch 1952; Greenspan 1983) already serve this function; others are to be presented here.

The batch settling of an initially uniform mixture in a centrifugal force field is considered anew. A general result concerning vorticity in the mixture allows reduction of the problem for 'long' containers of arbitrary cross-section to the determination of an analytic function. Separation in sectioned centrifuges is one application of potential technological significance.

2. Formulation

A fairly elaborate version of mixture theory currently used to study rotating, two-phase flows of particles and fluid is as follows:

Conservation of mass:

$$\frac{\partial \rho}{\partial t}+\nabla \cdot \rho \boldsymbol{v}=0, \tag{2.1}$$

where

$$\rho=\rho_{\mathrm{C}}(1-\alpha)+\rho_{\mathrm{D}} \alpha,$$

α is the volume fraction (and subscripts C, D refer to the continuous and discrete phases);

Conservation of dispersed phase:

$$\frac{\partial \alpha}{\partial t}+\nabla \cdot \alpha\left(\boldsymbol{v}+\frac{(1-\alpha) \rho_{\mathrm{C}}}{\rho} \boldsymbol{v}_{\mathrm{R}}\right)=0; \tag{2.2}$$

Conservation of momentum:

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t}+\tfrac{1}{2} \nabla \boldsymbol{v} \cdot \boldsymbol{v}+(\nabla \times \boldsymbol{v}) \times \boldsymbol{v}+2 \boldsymbol{\Omega} \times \boldsymbol{v}+\boldsymbol{\Omega} \times(\boldsymbol{\Omega} \times \boldsymbol{r})\right)$$

$$=-\nabla P+\nabla \cdot \boldsymbol{\pi}-\nabla \cdot \frac{\alpha(1-\alpha) \rho_{\mathrm{C}} \rho_{\mathrm{D}}}{\rho} \boldsymbol{v}_{\mathrm{R}} \boldsymbol{v}_{\mathrm{R}}; \tag{2.3}$$

Constitutive laws:

$$\boldsymbol{\pi}=\mu(\alpha)\left(\nabla \boldsymbol{v}+(\nabla \boldsymbol{v})^{+}\right)+\gamma(\alpha) \nabla \cdot \boldsymbol{v}\, \boldsymbol{I}; \tag{2.4}$$

$$\boldsymbol{v}_{\mathrm{R}}+\frac{\rho_{\mathrm{D}} \beta}{\rho_{\mathrm{C}} \Omega D(\alpha)(1+\epsilon \alpha)}\left(\frac{\partial \boldsymbol{v}_{\mathrm{R}}}{\partial t}+2 \boldsymbol{\Omega} \times \boldsymbol{v}_{\mathrm{R}}\right)$$

$$=-\frac{\beta}{D(\alpha)} \frac{\rho_{\mathrm{D}}-\rho_{\mathrm{C}}}{\rho_{\mathrm{C}}}(1-\alpha)\left[\frac{\partial \boldsymbol{v}}{\partial t}+\tfrac{1}{2} \nabla \boldsymbol{v} \cdot \boldsymbol{v}+(\nabla \times \boldsymbol{v}) \times \boldsymbol{v}+2 \boldsymbol{\Omega} \times \boldsymbol{v}+\boldsymbol{\Omega} \times(\boldsymbol{\Omega} \times \boldsymbol{r})\right]; \tag{2.5}$$

where $D(\alpha)$ and β are given below. The theory may be cast in a dimensionless form that is appropriate for a variety of problems by the scaling rules, $\boldsymbol{r} \rightarrow l \boldsymbol{r}$, $t \rightarrow t / \Omega R_{o}$, $\boldsymbol{v} \rightarrow R_{o} \Omega l \boldsymbol{v}$, $\boldsymbol{v}_{\mathrm{R}} \rightarrow|\epsilon| \beta \Omega l \boldsymbol{v}_{\mathrm{R}}$, $p \rightarrow \rho_{\mathrm{C}} \Omega^{2} l^{2}\left(R_{o} p+\frac{1}{2} r^{2}\right)$. The governing equations can then be written as

$$\nabla \cdot \boldsymbol{v}=\frac{\epsilon|\epsilon| \beta}{R_{o}} \nabla \cdot \frac{\alpha(1-\alpha)}{1+\epsilon \alpha} \boldsymbol{v}_{\mathrm{R}}, \tag{2.6}$$

where (2.1) and (2.2) have been combined;

$$\frac{\partial \alpha}{\partial t}+\nabla \cdot \alpha \boldsymbol{v}=-\frac{|\epsilon| \beta}{R_{o}} \nabla \cdot \frac{\alpha(1-\alpha)}{1+\epsilon \alpha} \boldsymbol{v}_{\mathrm{R}}; \tag{2.7}$$

$$(1+\epsilon \alpha)\left[R_{o}\left(\frac{\partial \boldsymbol{v}}{\partial t}+\tfrac{1}{2} \nabla \boldsymbol{v} \cdot \boldsymbol{v}+(\nabla \times \boldsymbol{v}) \times \boldsymbol{v}\right)+2 \hat{\boldsymbol{k}} \times \boldsymbol{v}\right]$$

$$=-\nabla p+E \nabla \cdot \boldsymbol{\pi}-\frac{\beta^{2} \epsilon^{2}}{R_{o}}(1+\epsilon) \nabla \cdot \frac{\alpha(1-\alpha)}{1+\epsilon \alpha} \boldsymbol{v}_{\mathrm{R}} \boldsymbol{v}_{\mathrm{R}}-\frac{\epsilon \alpha}{R_{o}} \hat{\boldsymbol{k}} \times(\hat{\boldsymbol{k}} \times \boldsymbol{r}); \tag{2.8}$$

$$\boldsymbol{v}_{\mathrm{R}}+\frac{(1+\epsilon) \beta(1-\alpha)}{D(\alpha)(1+\epsilon \alpha)}\left(R_{o} \frac{\partial \boldsymbol{v}_{\mathrm{R}}}{\partial t}+2 \hat{\boldsymbol{k}} \times \boldsymbol{v}_{\mathrm{R}}\right)$$

$$=-\frac{(1-\alpha) s}{D(\alpha)}\left[R_{o}^{2}\left(\frac{\partial \boldsymbol{v}}{\partial t}+\tfrac{1}{2} \nabla \boldsymbol{v} \cdot \boldsymbol{v}+(\nabla \times \boldsymbol{v}) \times \boldsymbol{v}\right)+2 R_{o} \hat{\boldsymbol{k}} \times \boldsymbol{v}+\hat{\boldsymbol{k}} \times(\hat{\boldsymbol{k}} \times \boldsymbol{r})\right]. \tag{2.9}$$

Here the Rossby number R_o is selected to characterize the velocity deviation from rigid rotation, and

$$\epsilon = \frac{(\rho_D - \rho_C)}{\rho_C}; \quad s = \frac{\rho_D - \rho_C}{|\rho_D - \rho_C|}; \quad \beta = \frac{2}{9}\frac{\Omega a^2}{\nu_C}; \quad E = \frac{\nu_C}{\Omega l^2}. \tag{2.10}$$

(In a typical application, $\epsilon = 0.01$, $a = 0.001$ cm, $\nu_C = 0.01$ cm²/s, $\Omega = 500\ \mathrm{s}^{-1}$, $l = 10$ cm, $\beta = 0.01$, $E = 2 \times 10^{-7}$. The particle Taylor number, β, though usually very small can be made $O(1)$ by controlling Ω in which case interesting phenomena can be expected.)

An empirical formula for the viscosity coefficient that is typical of slow, relative motions between phases is as given by Ishii & Chawla (1979)

$$D(\alpha) = \left(1 - \frac{\alpha}{\alpha_M}\right)^{-2.5\alpha_M}, \tag{2.11}$$

The accuracy of this formula requires inertial effects to be relatively unimportant and it is therefore consistent to neglect terms $O(R_o^2)$ on the right-hand side of (2.9) since R_o is itself small. Indeed, the equation so obtained is still more general than any used to date, perhaps too general given the state of experimental and theoretical knowledge of rapidly rotating mixtures. Terms multiplied by R_o could also be discarded for the same reason but are kept for the time being in order to satisfy prescribed initial conditions and to account qualitatively for a decrease in centrifugal force due to a retrograde rotation of the fluid. Therefore, the constitutive law for the relative velocity is taken to be

$$\boldsymbol{v}_R + \frac{(1+\epsilon)\beta(1-\alpha)}{D(\alpha)(1+\epsilon\alpha)}\left(R_o \frac{\partial \boldsymbol{v}_R}{\partial t} + 2\hat{\boldsymbol{k}} \times \boldsymbol{v}_R\right) = -\frac{(1-\alpha)s}{D(\alpha)}[2R_o(\hat{\boldsymbol{k}} \times \boldsymbol{v}) + \hat{\boldsymbol{k}} \times (\hat{\boldsymbol{k}} \times \boldsymbol{r})], \tag{2.12}$$

a form which embodies the essential physics, allows some exploration of new effects and does not add very much complexity to the theoretical treatment.

Initially the mixture is assumed to be uniform and quiescent so that

$$\alpha|_{t=0} = \alpha_0(\text{a constant}), \quad \boldsymbol{v}(\boldsymbol{r}, 0) = 0 = \boldsymbol{v}_R(\boldsymbol{r}, 0). \tag{2.13}$$

The usual viscous-fluid boundary condition, $\boldsymbol{v} = 0$, is imposed on the surface of the container, but it must be kept in mind that particles of finite size may indeed slip or roll on a solid wall, and discontinuities (kinetic shocks) will develop to separate the mixture from purified fluid or accumulated sediment.

The batch separation problem consists then of equations (2.6)–(2.8) and (2.12) subject to the foregoing initial and boundary conditions.

3. Vorticity in a rotating mixture

The equation for mixture vorticity,

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v},$$

obtained from (2.8), is a rather formidable expression:

$$(1+\epsilon\alpha)\, R_o \underline{\left(\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{v})\right)} + \epsilon R_o \nabla\alpha \times \left(\frac{\partial \boldsymbol{v}}{\partial t} + \tfrac{1}{2}\nabla \boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v}\right)$$

$$+ 2\epsilon \nabla\alpha \times (\hat{\boldsymbol{k}} \times \boldsymbol{v}) + \underline{2(1+\epsilon\alpha)\,\hat{\boldsymbol{k}}\nabla\cdot\boldsymbol{v}} - 2(1+\epsilon\alpha)\frac{\partial \boldsymbol{v}}{\partial z}$$

$$= E\nabla \times \nabla\cdot\boldsymbol{\pi} - \frac{\beta^2\epsilon^2}{R_o}(1+\epsilon)\,\nabla\times\nabla\cdot\frac{\alpha(1-\alpha)}{1+\epsilon\alpha}\boldsymbol{v}_{\mathrm{R}}\boldsymbol{v}_{\mathrm{R}} - \frac{\epsilon r}{R_o}\hat{\boldsymbol{r}}\times\nabla\alpha. \tag{3.1a}$$

A more useful formulation is in terms of the absolute vorticity $\boldsymbol{\zeta} = 2\hat{\boldsymbol{k}} + R_o\,\boldsymbol{\omega}$ and the total pressure $P = p + r^2/2R_o$, in which case it can be shown that

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\left(\frac{\boldsymbol{\zeta}}{1+\epsilon\alpha}\right) - \frac{\boldsymbol{\zeta}\cdot\nabla\boldsymbol{v}}{1+\epsilon\alpha} = \tfrac{1}{2}\nabla\frac{1}{(1+\epsilon\alpha)^2}\times\nabla P + \frac{E}{1+\epsilon\alpha}\nabla\times\frac{\nabla\cdot\boldsymbol{\pi}}{1+\epsilon\alpha}$$
$$- \frac{(1+\epsilon)}{(1+\epsilon\alpha)}\frac{\beta^2\epsilon^2}{R_o}\nabla\times\frac{1}{1+\epsilon\alpha}\nabla\cdot\frac{\alpha(1-\alpha)}{1+\epsilon\alpha}\boldsymbol{v}_{\mathrm{R}}\boldsymbol{v}_{\mathrm{R}}. \tag{3.1b}$$

Besides the usual vorticity generation mechanisms of line stretching, tilting and diffusion, the basic compressibility of a two-phase system, described by the volume fraction α, introduces baroclinic processes that are not often considered in mixture theory, although they constitute an essential aspect of dynamic meteorology.

The spatial distribution and time variation of the particle volume fraction produces vorticity and a velocity field, the magnitude of which is scaled by an appropriate choice of the Rossby number R_o. For example, in problems of batch settling from a uniform state, the balance of the first and fourth terms in (3.1*a*) (underlined) implies that $R_o = O(\epsilon)$. However, if at time zero, α is a definite function of position then more typically $R_o = O(|\epsilon|^{\frac{1}{2}})$ provided $\alpha_\theta|_{t=0} \neq 0$. An initial azimuthal dependency causes significant generation of baroclinic vorticity while for a purely radial variation, $\alpha|_{t=0} = \alpha_0(r)$, $R_o = O(|\epsilon|)$ once again. The transient evolutions are also very different in these cases. When α is not initially a constant, the volume fraction can develop a pronounced azimuthal stratification even though it may have none to start with; if α is a constant at the outset then α is strictly a time-dependent function thereafter. These matters are discussed in detail elsewhere, (Dahlkild & Greenspan 1987); in this work the focus is mainly on batch settling.

Consider separation in a 'long' cylindrical centrifuge of arbitrary cross-section with flat endplates. In this approximation, the secondary $O(E^{\frac{1}{2}})$ flows induced by the endwall Ekman layers are small compared to the circulations produced directly by the centrifugal force. A relative measure of the mechanisms is given by

$$\lambda = E^{\frac{1}{2}}/|\epsilon|\beta H, \tag{3.2}$$

which is the ratio of the particle settling time to the viscous spin-up time in the centrifuge; 'long' implies $\lambda \ll 1$. (Here H is the aspect ratio of the cylinder, i.e. the length divided by the radius.)

The initial conditions for batch settling are $\alpha = \alpha_0$, a constant, and $\boldsymbol{v} = 0$ at time zero. The solution of the equations of motion within the mixture domain follows from the recognition that subsequently

$$\alpha = \alpha(t), \tag{3.3}$$

$$\hat{\boldsymbol{k}} \cdot \boldsymbol{v} = 0, \quad \frac{\partial \boldsymbol{v}}{\partial z} = 0, \tag{3.4}$$

$$\nabla \times \boldsymbol{v} = \boldsymbol{\omega} = \omega(t)\,\hat{\boldsymbol{k}}, \quad \nabla \times \boldsymbol{v}_{\mathrm{R}} = \boldsymbol{\omega}_{\mathrm{R}} = \omega_{\mathrm{R}}(t)\,\hat{\boldsymbol{k}}, \tag{3.5}$$

$$\nabla \cdot \boldsymbol{v} = f(t), \quad \nabla \cdot \boldsymbol{v}_{\mathrm{R}} = f_{\mathrm{R}}(t). \tag{3.6}$$

The velocity is two-dimensional and independent of the vertical coordinate. The substitution of these forms in (2.6) and (2.7) gives

$$f(t) = \frac{s\epsilon^2\beta}{R_o}\frac{\alpha(1-\alpha)}{1+\epsilon\alpha} f_{\mathrm{R}}(t), \tag{3.7}$$

and

$$\alpha'(t) = -\frac{|\epsilon|\beta}{R_o}\alpha(1-\alpha) f_{\mathrm{R}}(t) = -\frac{1+\epsilon\alpha}{\epsilon} f(t). \tag{3.8}$$

The vertical component of the vorticity equation (3.1 *a*) implies

$$R_o[\omega'(t) + \omega(t)f(t)] + 2f(t) = -2\frac{\beta^2\epsilon^2(1+\epsilon)\,\alpha(1-\alpha)}{R_o(1+\epsilon\alpha)^2} f_{\mathrm{R}}(t)\,\omega_{\mathrm{R}}(t). \tag{3.9}$$

Lastly, the divergence and curl of (2.12) yield

$$f_{\mathrm{R}}(t) + \frac{(1+\epsilon)\,\beta(1-\alpha)}{D(\alpha)\,(1+\epsilon\alpha)}(R_o f'_{\mathrm{R}}(t) - 2\omega_{\mathrm{R}}(t)) = 2\frac{1-\alpha}{D(\alpha)} s(1 + R_o\,\omega(t)); \tag{3.10}$$

$$\omega_{\mathrm{R}}(t) + \frac{(1+\epsilon)\,\beta(1-\alpha)}{D(\alpha)\,(1+\epsilon\alpha)}(R_o\,\omega'_{\mathrm{R}}(t) + 2f_{\mathrm{R}}(t)) = -2\frac{1-\alpha}{D(\alpha)} sR_o f(t). \tag{3.11a}$$

Equation (2.9), the more exact constitutive law for the relative velocity $\boldsymbol{v}_{\mathrm{R}}$, does now allow $\nabla \cdot \boldsymbol{v}_{\mathrm{R}}$ to be expressed as a function of time alone. However it does provide a formula for $\omega_{\mathrm{R}}(t)$ which, by using (3.9), can be written as:

$$\omega_{\mathrm{R}}(t) + \frac{(1+\epsilon)\,\beta(1-\alpha)}{(1+\epsilon\alpha)\,D(a)}(R_o\,\omega'_{\mathrm{R}}(t) + 2f_{\mathrm{R}}(t)) = \frac{2s\beta^2\epsilon^2(1+\epsilon)\,\alpha(1-\alpha)^2}{(1+\epsilon\alpha)^2 D(\alpha)} f_{\mathrm{R}}(t)\,\omega_{\mathrm{R}}(t). \tag{3.11b}$$

The $O(R_o^2)$ discrepancy between the two equations for $\omega_{\mathrm{R}}(t)$, (3.11 *a, b*) is indicative of the error involved when inertial terms are neglected in the rule for the relative velocity.

There are five equations for the five variables $\alpha(t)$, $f(t)$, $f_{\mathrm{R}}(t)$, $\omega(t)$, $\omega_{\mathrm{R}}(t)$, and the system is well posed and readily integrated subject to the initial conditions

$$\alpha(0) = \alpha_0, \quad f(0) = f_{\mathrm{R}}(0) = \omega(0) = \omega_{\mathrm{R}}(0) = 0. \tag{3.12}$$

Moreover since time does not appear explicitly, four of the unknowns can be written as functions of the fifth, the natural choice being

$$f(\alpha), \quad f_{\mathrm{R}}(\alpha), \quad \omega(\alpha), \quad \omega_{\mathrm{R}}(\alpha);$$

(3.8) is then used to find $\alpha(t)$.

Integration of the vorticity equation (3.9) (or more directly from (3.1*b*)) yields

$$\omega = \frac{2\epsilon}{R_o}\frac{\alpha-\alpha_0}{1+\epsilon\alpha_0} + \frac{2\beta|\epsilon|}{R_o}(1+\epsilon)(1+\epsilon\alpha)\int_{\alpha_0}^{\alpha}\frac{\omega_{\mathrm{R}}(\alpha)}{(1+\epsilon\alpha)^3}\,\mathrm{d}\alpha, \tag{3.13}$$

which expresses the production of potential vorticity $\hat{\boldsymbol{k}}\cdot\boldsymbol{\zeta}/1+\epsilon\alpha$. In practice, parameters ϵ, β and R_o are usually small (in varying degrees), and the integral in (3.13), which originates from the diffusion stress tensor, is readily shown to be $O(\beta^2\epsilon/R_o)$ and therefore of negligible importance in most circumstances. A simpler formula for relative vorticity is then a consequence of the strict conservation of potential vorticity:

$$\omega = \frac{2\epsilon}{R_o}\frac{\alpha-\alpha_0}{1+\epsilon\alpha_0}. \tag{3.14}$$

At present this is probably as accurate as any other given the lack of experimental information about a rotating mixture with β moderate, and the *ad hoc* nature of the constitutive law for the relative velocity.

Estimates of the magnitudes of the primary variables are

$$\omega = O\left(\frac{|\epsilon|}{R_o}\right),\quad \omega_{\mathrm{R}} = O(\beta),\quad f = O\left(\frac{\epsilon^2\beta}{R_o}\right),\quad f_{\mathrm{R}} = O(1),$$

and clearly the choice $R_o = |\epsilon|$ (or $R_o = |\epsilon|\alpha_0$ if α_0 is also very small) properly characterizes the velocity field in these problems of batch separation. Examination of the system of equations reveals that except for a very short, $O(R_o\beta)$, time transient when adjustments to the initial conditions are made, the time derivatives in (3.10) and (3.11) can be neglected, which allows f_{R} and ω_{R} to be determined algebraically. (These time derivatives probably have little physical significance anyway and were included until now to construct a well-posed initial-value problem. At best the incorporation of these terms might be indicative of processes not yet recognized.) It follows that, to $O(R_o^2)$,

$$f_{\mathrm{R}} \approx 2sk\left(\frac{1-\alpha}{D(\alpha)}\right)(1+R_o\omega),\quad \omega_{\mathrm{R}} \approx -\frac{2(1-\alpha)\beta}{D(\alpha)(1+\epsilon\alpha)}(1+\epsilon)f_{\mathrm{R}}, \tag{3.15}$$

where
$$k \approx \left[1+\frac{4(1+\epsilon)^2\beta^2(1-\alpha)^2}{D^2(\alpha)(1+\epsilon\alpha)^2}\right]^{-1}.$$

The first of these, together with (3.14), may be used to rewrite (3.8) as an equation for α alone. Integration to find $\alpha(t)$ allows $\omega_{\mathrm{R}}, \omega, f_{\mathrm{R}}, f$ to be determined since each has been written solely and explicitly as a function of the volume fraction. If all parameters are small and second-order quantities $\beta^2, \epsilon^2\epsilon\beta$ are neglected, the result is

$$\alpha' \approx -\frac{2\beta s\alpha(1-\alpha)^2}{D(\alpha)}, \tag{3.16}$$

the integral of which must be still obtained numerically. Since the velocity field does not produce a spatial variation of the volume fraction in problems of bulk separation, the timescale here is in units of $1/\beta$ or $1/\epsilon\beta\Omega$, dimensionally. This is just the characteristic time for a particle to settle in a centrifugal force field.

If the mixture is also dilute $\alpha \ll 1$, and $\rho_{\mathrm{D}} > \rho_{\mathrm{C}}$ or $s = 1$, then (3.16) is easily integrated:

$$\alpha = \alpha_0\,\mathrm{e}^{-2\beta t}, \tag{3.17}$$

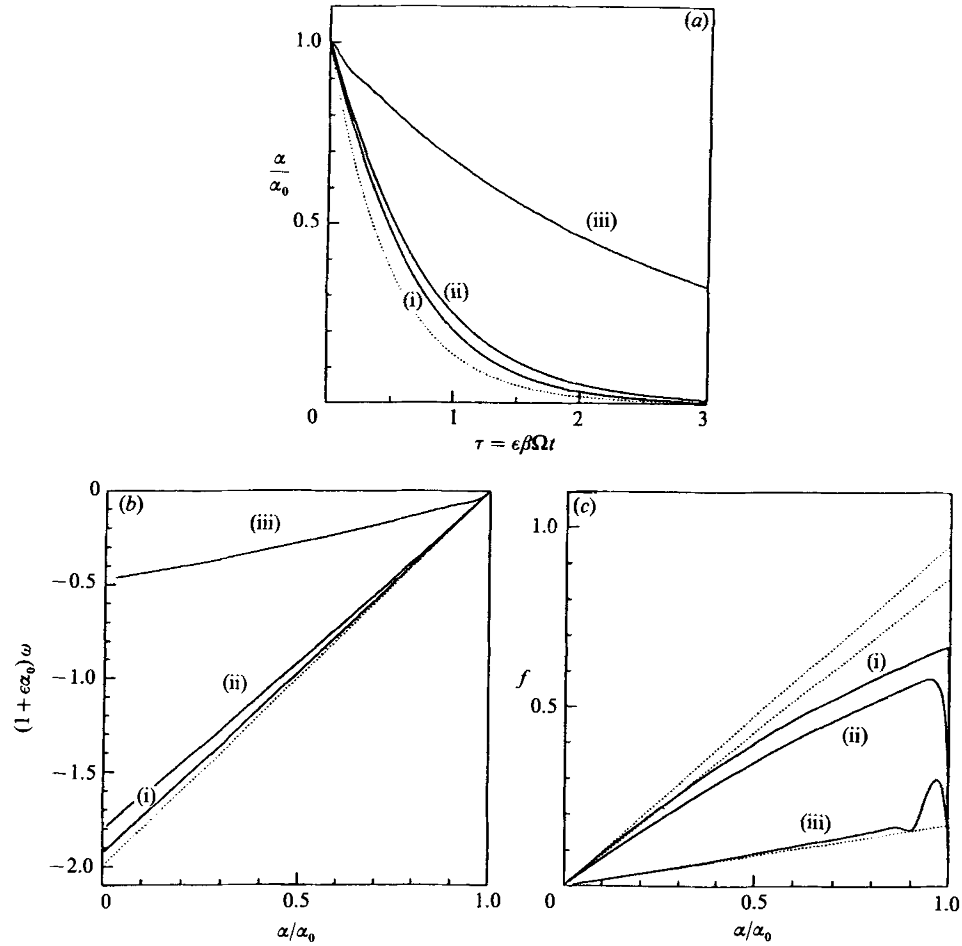

FIGURE 1. (*a*) Volume fraction α versus time, with (*b*) ω, and (*c*) f versus α for typical values of the basic parameters $\alpha_0 = 0.05$, and $R_o = \epsilon\alpha_0$: (i) $\epsilon = 0.1$, $\beta = 0.01$; (ii) $\epsilon = 1$, $\beta = 0.1$; (iii) $\epsilon = 0.1$, $\beta = 1$.

so that

$$\omega = -\frac{2\alpha_0}{1+\epsilon\alpha_0}(1-e^{-2\beta t}), \tag{3.18}$$

$$f = \frac{2\epsilon^2\beta\alpha_0}{R_o}e^{-2\beta t}. \tag{3.19}$$

A negative relative vorticity is produced in the separation process which in the axisymmetric container is equivalent to the retrograde rotation found previously. Equations (3.14), (3.8) and (3.15) imply more generally that a negative relative vorticity is *always* produced in the separation process, whether the density difference is positive or negative. The vorticity deficit is eventually adjusted by the ordinary viscous spin-up process of a homogeneous fluid which occurs on the $O(E^{-\frac{1}{2}})$ timescale.

Comparisons of the exact numerical solution and the simpler approximate formulas for volume fraction (3.17), vorticity (3.14), and divergence (3.7) (with $f_R = 2$) are shown in figure 1 for the representative values $\alpha_0 = 0.05$, $\epsilon = 0.1$, 1, $\beta = 0.1$, 1 and $R_o = \epsilon\alpha_0$. As anticipated, the deviation is appreciable only for β large.

4. The velocity field

The equations of motion governing bulk separation of an initially homogeneous mixture in a 'long' cylinder reduce to

$$\nabla \cdot \boldsymbol{v} = f(\alpha), \tag{4.1}$$

$$\nabla \times \boldsymbol{v} = \omega(\alpha)\,\hat{\boldsymbol{k}}, \tag{4.2}$$

where the mass-averaged velocity is a two-dimensional vector, $\hat{\boldsymbol{k}}\cdot\boldsymbol{v} = 0$, that is independent of vertical distance, $\partial \boldsymbol{v}/\partial z = 0$, and functions $f(\alpha)$, $\omega(\alpha)$, $\alpha(t)$ are known. If a new vector function $\boldsymbol{q}$ is defined by

$$\boldsymbol{v} = \boldsymbol{q} + \tfrac{1}{2}r(f(\alpha)\,\hat{\boldsymbol{r}} + \omega(\alpha)\,\hat{\boldsymbol{\theta}}), \tag{4.3}$$

then it follows that

$$\nabla \cdot \boldsymbol{q} = 0, \tag{4.4}$$

$$\nabla \times \boldsymbol{q} = 0. \tag{4.5}$$

Therefore

$$\boldsymbol{q} = \nabla \Phi = \nabla \times \Psi \hat{\boldsymbol{k}},$$

and

$$\mathscr{F} = \Phi + \mathrm{i}\Psi$$

is an analytic function of the complex variable $z = x + \mathrm{i}y$.

Let

$$\left.\begin{aligned} q_x &= \hat{\boldsymbol{i}}\cdot\boldsymbol{q}, \quad q_y = \hat{\boldsymbol{j}}\cdot\boldsymbol{q},\\ q_r &= \hat{\boldsymbol{r}}\cdot\boldsymbol{q}, \quad q_\theta = \hat{\boldsymbol{\theta}}\cdot\boldsymbol{q}\end{aligned}\right\} \tag{4.6}$$

denote the components of (any) vector $\boldsymbol{q}$ in Cartesian and cylindrical coordinates. It follows that the complex-conjugate velocity

$$\overline{W} = q_x - \mathrm{i}q_y = \frac{\partial \Phi}{\partial x} + \mathrm{i}\frac{\partial \Psi}{\partial x} = \frac{\mathrm{d}\mathscr{F}}{\mathrm{d}z} \tag{4.7}$$

is also an analytic function, but $v_x - \mathrm{i}v_y$ is not because by definition

$$v_x - \mathrm{i}v_y = \tfrac{1}{2}(f(\alpha) - \omega(\alpha)\,\mathrm{i})\bar{z} - q_x - \mathrm{i}q_y. \tag{4.8}$$

It is also convenient to express the last equation in terms of the velocity components in cylindrical coordinates. Since

$$v_r - \mathrm{i}v_\theta = (v_x - \mathrm{i}v_y)\,\mathrm{e}^{\mathrm{i}\theta} \tag{4.9}$$

the result is

$$v_r - \mathrm{i}v_\theta = \tfrac{1}{2}\,\mathrm{e}^{\mathrm{i}\theta}(f(\alpha) - \omega(\alpha)\,\mathrm{i})\,\bar{z} + q_r - \mathrm{i}q_\theta \tag{4.10}$$

and

$$r(q_r - \mathrm{i}q_\theta) = r\,\mathrm{e}^{\mathrm{i}\theta}\overline{W} = z\overline{W}(z). \tag{4.11}$$

With $Q = rq_r$, and $\Gamma = rq_\theta$, the last equation shows that $Q - \mathrm{i}\Gamma$ is also an analytic function of z.

The problem of batch settling in a centrifugal force field that was formulated in §2 has now been reduced to the determination of an analytic function $\overline{W}(z)$ which satisfies certain prescribed boundary conditions. The specification of these conditions is difficult because the region occupied by the mixture, as distinct from that of sediment or purified fluid, changes in time. If, initially, the particle velocity $\boldsymbol{v}_{\mathrm{D}}$ is directed towards a solid boundary with outward unit normal $\hat{\boldsymbol{n}}$, $\hat{\boldsymbol{n}}\cdot\boldsymbol{v}_{\mathrm{D}} > 0$, then sediment accumulates there and a kinematic shock develops. Likewise, if $\hat{\boldsymbol{n}}\cdot\boldsymbol{v}_{\mathrm{D}} < 0$, a discontinuity is required between mixture and the purified continuous fluid phase that is adjacent to the wall. The loci of these shocks are an intrinsic part of the separation process. The proper procedure is to require $\boldsymbol{v} \equiv 0$ at the container wall, or $\boldsymbol{v}\cdot\hat{\boldsymbol{n}} = 0$ for an inviscid fluid, and then to determine the position of any and all interfaces by utilizing the shock conditions for mass and momentum. However, this is a very hard programme to implement unless there is complete axial symmetry (Greenspan 1983) or the interfaces actually remain close enough to the boundaries to be indistinguishable from them. This can happen when the mixture is dilute and there is a minimal amount of sediment or the purified fluid moves mainly in a boundary layer. In these situations, simple but accurate approximations introduced by Schneider (1982), were based on the assumptions that the thin sediment layer carries almost no volume flux, so that

$$\boldsymbol{j}\cdot\hat{\boldsymbol{n}} = \left(\boldsymbol{v} - \frac{\epsilon^2\beta}{R_o}\frac{\alpha(1-\alpha)}{1+\epsilon\alpha}\boldsymbol{v}_{\mathrm{R}}\right)\cdot\hat{\boldsymbol{n}} = 0. \tag{4.12}$$

on that part of the solid boundary, while

$$\boldsymbol{v}_{\mathrm{D}}\cdot\hat{\boldsymbol{n}} = \left(\boldsymbol{v} + \frac{\epsilon\beta}{R_o}\frac{1-\alpha}{1+\epsilon\alpha}\boldsymbol{v}_{\mathrm{R}}\right)\cdot\hat{\boldsymbol{n}} = 0, \tag{4.13}$$

wherever a boundary-layer current of pure fluid forms. Some of the shock positions are fixed this way, as one and the same with the solid boundaries. It is advantageous and consistent with the accuracy of these approximations to employ a somewhat simpler constitutive law (Greenspan & Ungarish 1985) because this allows explicit boundary conditions to be set on the mass-averaged velocity, $\boldsymbol{v}$:

$$\boldsymbol{v}_{\mathrm{R}} = s\frac{1-\alpha}{D(\alpha)}\left(1+\frac{4\beta^2(1-\alpha)^2(1+\epsilon)^2}{D^2}\right)^{-1} r\left(\hat{\boldsymbol{r}} - \frac{2(1-\alpha)\beta(1+\epsilon)}{D(\alpha)}\hat{\boldsymbol{\theta}}\right) \tag{4.14}$$

$$= rU_{\mathrm{R}}(\alpha)\,\hat{\boldsymbol{r}} + rV_{\mathrm{R}}(\alpha)\,\hat{\boldsymbol{\theta}} = \tfrac{1}{2}(f_{\mathrm{R}}\,\hat{\boldsymbol{r}} + \omega_{\mathrm{R}}\,\hat{\boldsymbol{\theta}}). \tag{4.15}$$

As an example of some technological interest, consider bulk separation in a long, sectioned, cylindrical centrifuge, the semicircular domain shown in figure 3. In particular, let the particles be heavier than the fluid in which they are suspended so that $s = 1$ (in which case no purified core of fluid develops in the full cylinder). The boundary condition (4.13) applies to the section $\theta = \pi$, whereas (4.12) is imposed on $\theta = 0$ and the circumference $r = a$. The analysis is much facilitated using cylindrical coordinates and components of velocity. In terms of the analytic function defined in (4.11),

$$Q - \mathrm{i}\Gamma = r(q_r - \mathrm{i}q_\theta) = z\bar{W}(z)$$

and the relative velocity as given in (4.15), the boundary conditions can be written as follows:

$$\text{on } \theta = 0, v_\theta = \frac{\epsilon^2\beta}{R_o}\frac{\alpha(1-\alpha)\,r}{1+\epsilon\alpha}V_{\mathrm{R}}(\alpha)$$

or

$$\Gamma = -A(\alpha)\,r^2 = -r^2\left(\tfrac{1}{2}\omega(\alpha) - \frac{\epsilon^2\beta}{R_o}\frac{(1-\alpha)\,\alpha}{1+\epsilon\alpha}V_{\mathrm{R}}(\alpha)\right); \tag{4.16}$$

$$\text{on } r = a, \quad v_r = \frac{\epsilon^2\beta}{R_o}\frac{\alpha(1-\alpha)\,a}{1+\epsilon\alpha}U_{\mathrm{R}}(\alpha)$$

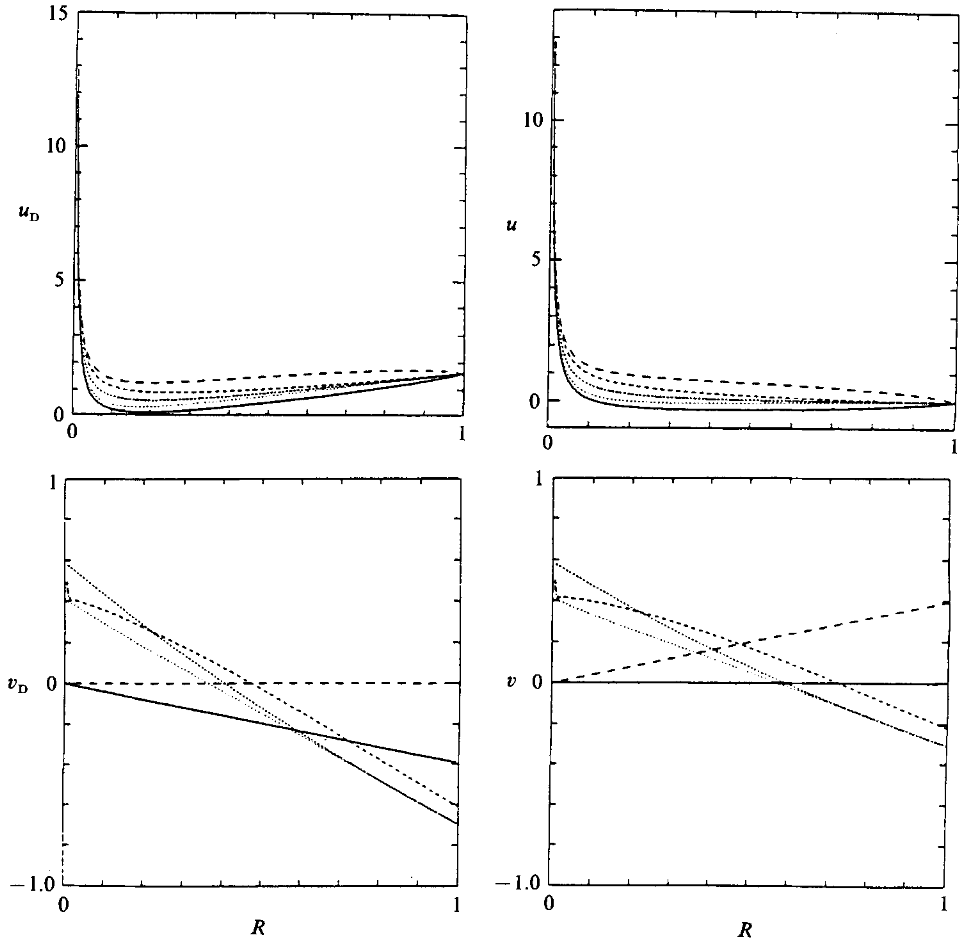

FIGURE 2. Particle and mass-centred velocity components versus radial distance along five rays: $\theta = 0, 45°, 90°, 135°, 180°$, with $\epsilon = 0.1$, $\beta = 0.1$, $\alpha_0 = 0.05$, $\alpha = 0.025$ and $R_o = \epsilon\alpha_0$. Radial velocity components are sequenced monotonically for increasing angle θ; the solid line corresponds to $\theta = 0$.

or
$$Q = C = a^2\left(\frac{\epsilon^2\beta}{R_o}\frac{a(1-\alpha)}{1+\epsilon\alpha}U_R(\alpha) - \tfrac{1}{2}f(\alpha)\right); \tag{4.17}$$

$$\text{on } \theta = \pi, \quad v_\theta = -\frac{\epsilon\beta}{R_o}\frac{1-\alpha}{1+\epsilon\alpha}rV_R(\alpha)$$

or
$$\Gamma = -B(\alpha)\,r^2 = -r^2\left(\tfrac{1}{2}\omega(\alpha) + \frac{\epsilon\beta}{R_o}\frac{1-\alpha}{1+\epsilon\alpha}V_R(\alpha)\right). \tag{4.18}$$

The function $Q - \mathrm{i}\Gamma$ must be found that is analytic in the semicircle and assumes the prescribed values shown on the segments of the bounding contour. With $\zeta = z/a$, this function is

$$Q - \mathrm{i}\Gamma = z\bar{W}(z) = \frac{1}{\pi}\left[(B-A)\,\zeta^2\log\zeta + \mathrm{i}\pi A\zeta^2 \right.$$
$$\left. + (B+A)\left(\zeta - \frac{1}{\zeta}\right) + \left(\zeta^2 - \frac{1}{\zeta^2}\right)(A\log(1-\zeta) - B\log(1+\zeta)\right] + C. \tag{4.19}$$

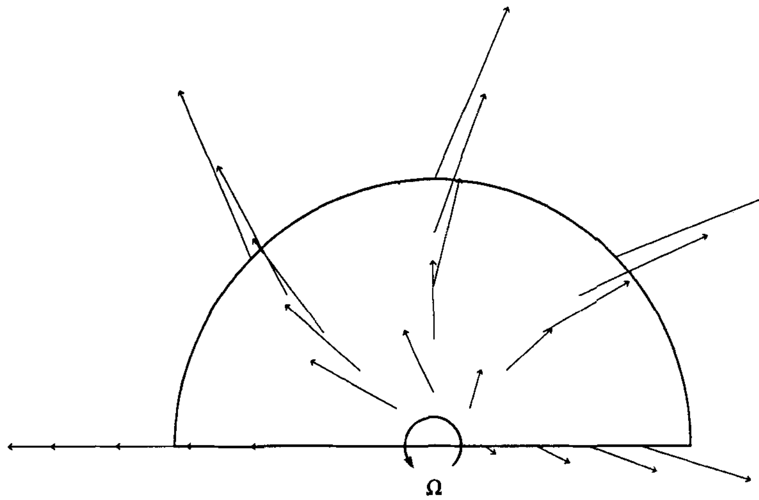

FIGURE 3. Particle velocity vectors at the parameter values of figure 2.

The components of the mass-averaged velocity are recovered from (4.11);

$$v_r - \mathrm{i}v_\theta = \tfrac{1}{2}r(f(\alpha) - \omega(\alpha)\,\mathrm{i}) + \frac{1}{r}(Q - \mathrm{i}\Gamma), \tag{4.20}$$

and the definitions of $f(\alpha)$, $\omega(\alpha)$, A, B, and C. Of course, the formulas are complicated, but nonetheless, they are the explicit, exact solution of the original equations of motion with the boundary conditions imposed. Figure 2 shows the particle and mass-centred velocity components along five rays for the representative values $\epsilon = 0.1$, $\beta = 0.1$, $\alpha_0 = 0.05$, $\alpha = 0.025$. The azimuthal velocity component changes sign at about the midcircle position, but particles are always directed outwards as shown in figure 3. A weak source if evident at the origin. The $O(\beta)$-flow in the boundary layer that must be a consequence of the volume-flux condition (4.13) implies the formation and growth of a core of purified fluid, but this aspect is not considered here!

Procedures exist to obtain solutions in different domains but the results are not usually in a form that involves only simple functions. With different values for A, B and C, (4.19) also gives the solution to the problem with, say $\boldsymbol{v}\cdot\hat{\boldsymbol{n}} = 0$ on the periphery. This enables the effects of particular conditions to be assessed and compared. These and other matters will be discussed elsewhere.

5. Conclusion

The vorticity produced in the bulk separation of an initially homogeneous mixture is strongly linked to the volume fraction of the dispersed phase. If endwall effects are neglected, the vorticity in the mixture is a simple function of the volume fraction, α, which itself depends only on time. This allows reduction of the complex governing equations for this class of motions to a two-dimensional potential-flow problem whose exact solution can even be given in closed form in certain configurations. Any occurrence and propagation of a kinematic shock that separates purified fluid from mixture makes the solution more difficult to obtain because the locus of the jump discontinuity is essential and this calculation is intrinsically nonlinear in character.

The postulated form of solution equations (3.3)–(3.6) also satisfies the viscous-stress terms, $E\nabla \times \nabla \cdot \boldsymbol{\pi}$, but fails if advection $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$ is included in the constitutive law for the relative velocity $\boldsymbol{v}_{\mathrm{R}}$. These were the $O(R_o^2)$ quantities dropped at the outset.

Endwall effects of the Ekman layers can be incorporated in the theory (Dahlkild and Greenspan 1987), the analysis of which then reverts to a perturbation procedure whose lowest-order approximation for the interior flow is essentially that given here.

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